8.3-8.4: Trigonometric Integrals and Substitution

Powers of Sine and Cosine:

$$\int \sin^m x \cos^n x dx$$

- **Case 1:** If m is odd, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

  $$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x.$$

  We then combine the single $\sin x$ with the dx and set $\sin x dx$ equal to $-d(\cos x)$.

- **Case 2:** If m is even and n is odd, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

  $$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

  We then combine the single $\cos x$ with the dx and set $\cos x dx$ equal to $d(\sin x)$.

- **Case 3:** If m and n are even, we substitute

  $$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

  to reduce the integrand to one in lower powers of $\cos 2x$.

1. $\int \sin^3 x \cos^2 x dx$
2. $\int \cos^5 x dx$
3. $\int \sin^2 x \cos^4 x dx$

Eliminating Square Roots: We can use the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ to evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} dx.$$

Powers of Tangent and Secant: We can use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$ to evaluate certain powers $\int \tan^m x \sec^n x dx$.

1. $\int \tan^4 x dx$
2. $\int \sec^3 x dx$
3. $\int \tan^4 x \sec^4 x dx$

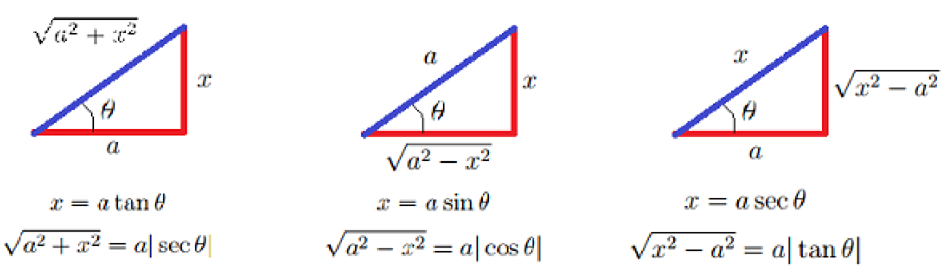
Products of Sines and Cosines: We can use the Product-to-Sum identities to evaluate

$$\int \sin mx \sin nx dx, \quad \int \sin mx \cos nx dx, \text{ and } \quad \int \cos mx \cos nx dx.$$

1. $\int \sin 3x \cos 5x dx$
2. $\int \sin 3x \sin 3x dx$
3. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx$
4. $\int \sin\theta \cos\theta \cos 3\theta d\theta$
5. $\int \cos^2 2\theta \sin\theta d\theta$

4 and 5 above require the use of multiple trigonometric identities to solve.

Trigonometric Substitutions: The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$ and $x = a\sec\theta$. Along the Pythagorean identities, we can use these to transform integrals with $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$ into integrals we can evaluate. We then use reference triangles to transform back to our original variable.



1. $\int \frac{dx}{\sqrt{4+x^2}}$
2. $\int \frac{dx}{\sqrt{a^2+x^2}}$
3. $\int \frac{x^2 dx}{\sqrt{9-x^2}}$
4. $\int \frac{dx}{\sqrt{25x^2-4}}, \qquad x > \frac{2}{5}$